International Journal of Theoretical Physics, Vol. 39, No. 2, 2000

Analytic Advantages of Spherically Symmetric Step-Function Potentials in the Dirac Equation with Scalar and Fourth Component of Vector Potential

G. J. Papadopoulos,[1] C. G. Koutroulos,[1] and M. E. Grypeos[1]

Received June 10, 1999

The root mean square radii of the particle orbits are calculated (semi)analytically for every bound state, using the Dirac equation with a scalar potential U_S and fourth component of a vector potential U_V in the case of a spherically symmetric step-function shape with the same radius R for these potentials. In addition, a (semi)analytic expression of the expectation value of the corresponding potential energy operator is derived. For the above quantities, expressions of the energy eigenvalues in terms of the potential parameters are needed and approximate formulas may be used in certain cases. This study emphasizes the analytic advantages of the relativistic, spherically symmetric step-function potential model. Its applicability is discussed in connection with a problem of physical interest, namely that of the motion of a Λ particle in hypernuclei.

1. INTRODUCTION

As is very well known, the spherically symmetric step-function potential well is one of the potentials for which the corresponding eigenvalue problem in nonrelativistic quantum mechanics can be solved "semianalytically" for every bound state, that is, the energy eigenfuctions are given analytically in terms of well-known functions, while for the corresponding eigenvalues one has to solve numerically a transcendental equation [1–3]. Nevertheless, approximate analytic expressions also can be derived for the energy eigenvalues in certain cases (see, for example, refs. 3–5). Because of its interesting

[1] Department of Theoretical Physics, Aristotle University of Thessaloniki, Greece.

analytic advantages, this potential has been widely used in applications in spite of its simplified shape.

During recent decades, considerable theoretical work has been done on the basis of the (generalized) Dirac equation instead of the Schrödinger equation (see, e.g., refs. 6–9 and references therein). In this equation, both a scalar potential $U_S(r)$ and the fourth component of a vector potential $U_V(r)$ appear which are attractive and repulsive, respectively. In the case of step-function shape, with the same radius R for those spherically symmetric potentials, a semianalytic treatment is possible for this eigenvalue problem, too [10–11]. The bound-state energy eigenvalues can be also given by approximate analytic expressions [12].

The present work aims to advance the detailed study of the relativistic, spherically symmetric step-function potential model. It emphasizes its analytic advantages for various bound states in estimating quantities of physical interest other than energy eigenvalues. It is shown, by extending previous results for the ground state [13], that an exact semianalytic expression can be derived for the root-mean-square radius of the particle orbit (in an energy eigenstate) $\langle r^2 \rangle^{1/2}$ for every bound state. Furthermore, an exact semianalytic expression for the corresponding expectation value of the potential energy operator $\langle V \rangle$ is also derived for every bound state. These expressions and mainly the first one are very complex. It turns out, however, that an approximate treatment of the previous results can be made and simpler approximate analytic expressions can be derived in certain cases.

It should be pointed out that even the nonrelativistic expressions of the root-mean-square radii of the particle orbits and of the expectation values of the potential energy operator for the various eigenstates (which follow from the relativistic expressions derived in this paper by taking the nonrelativistic limit) have not been given before, to our knowledge, with the exception of those referred to the ground state for $\langle r^2 \rangle^{1/2}$.

The arrangement of this paper is as follows: In the next section, the basic formulas used are exhibited and the derived analytic expressions for the above-mentioned quantities are given. First, the exact expressions are given and subsequently the approximate ones. In the final section, numerical results are given and discussed. To be more specific and close to a problem of physical interest, the notation used and the numerical results obtained refer to a Λ particle in hypernuclei, using reasonable values for the potential parameters. Λ-particle binding energies in certain single-particle states are known experimentally for a number of hypernuclei and some of them have been used for comparisons with the theoretical results. Finally, a discussion is made pertaining to the validity of the approximations and the estimate of the errors in connection with the range of the parameters.

2. BASIC FORMALISM AND ANALYTIC RESULTS

It is assumed that the average potential between the Λ particle and the nucleus is made up of an attractive scalar relativistic single-particle potential $U_S(r)$ and a repulsive relativistic single-particle potential $U_V(r)$ which is the fourth component of a vector potential, and that the differential equation describing the motion of the Λ particle in hypernuclei is the Dirac equation

$$[c\vec{\alpha} \cdot \vec{p} + \beta\mu c^2 + \beta U_S(r) + U_V(r)]\psi = E\psi \tag{1}$$

where $\vec{\alpha} = (\alpha_1, \alpha_2, \alpha_3)$, and β are the Dirac matrices. E is the total energy (i.e., $E = -B_\Lambda + \mu c^2$, B_Λ being the binding energy of the Λ particle) and ψ is the Dirac four-spinor (we are using the formalism outlined in, e.g., ref. 10 and references therein).

Instead of the potentials $U_S(r)$ and $U_V(r)$, we use the potentials

$$U_\pm(r) = U_S(r) \pm U_V(r) \tag{2}$$

which are both attractive. We consider the case in which $U_+(r)$ and $U_-(r)$ are spherically symmetric step-function wells having the same radius R and depths D_+ and D_-, respectively, i.e.,

$$U_\pm(r) = -D_\pm[1 - \Theta(r - R)] \tag{3}$$

where Θ is the unit step function and $R = r_0 A_c^{1/3}$. Here A_c is the mass number of the core system. In such a case the generalized Dirac equation may be solved "semianalytically" for every bound state. We find it more convenient, however, to express the large and small component wave functions in terms of the spherical Bessel j_l and the spherical MacDonald functions k_l [14] instead of the spherical Hankel functions. The expressions of G and F can then be written

$$G(r) = \tilde{N}nr\left\{[1 - \Theta(r - R)]j_l(nr) + \Theta(r - R)\frac{j_l(nR)}{k_l(n_0 R)}k_l(n_0 r)\right\} \tag{4}$$

$$F(r) = \tilde{N}nc\hbar\left\{[1 - \Theta(r - R)]\frac{1}{-B_\Lambda + 2\mu c^2 - D_-}[nrj_{l-1}(nr) + (k - l)j_l(nr)]\right.$$
$$\left. + \Theta(r - R)\frac{1}{-B_\Lambda + 2\mu c^2}\frac{j_l(nR)}{k_l(n_0R)}[-n_0rk_{l-1}(n_0r) + (k - l)k_l(n_0r)]\right\} \tag{5}$$

while the energy eigenvalue equation becomes

$$-\left[1 - \frac{D_-}{2\mu c^2 - B_\Lambda}\right]\frac{n_0Rk_{l-1}(n_0R)}{k_l(n_0R)} = \frac{(k - l)D_-}{2\mu c^2 - B_\Lambda} + \frac{nRj_{l-1}(nR)}{j_l(nR)} \tag{6}$$

In these expressions, $k = \pm(j + 1/2)$, $j = (l \mp 1/2)$, and the quantities n and n_0 are defined as follows:

$$n = \left\{ \frac{2\mu}{\hbar^2} (D_+ - B_\Lambda)[1 - (D_- + B_\Lambda)(2\mu c^2)^{-1}] \right\}^{1/2} \tag{7}$$

$$n_0 = \left\{ \frac{2\mu}{\hbar^2} [B_\Lambda(1 - B_\Lambda(2\mu c^2)^{-1})] \right\}^{1/2} \tag{8}$$

The quantum numbers in G, F, B_Λ, and $\tilde{N}$ have been suppressed.

The normalization constant $\tilde{N}$ is calculated using the following normalization condition:

$$\int_0^\infty [G^2(r) + F^2(r)]\, dr = 1 \tag{9}$$

After using the expressions for the radial components $G(r)$ and $F(r)$ of the wavefunction given above, we find for the normalization constant the following formula:

$$\begin{aligned} \tilde{N} = \frac{1}{n} \Bigg\{ & \frac{j_l^2(nR)}{k_l^2(n_0R)} \frac{R^3}{2} k_{l-1}(n_0R) k_{l+1}(n_0R) - \frac{R^3}{2} j_{l-1}(nR) j_{l+1}(nR) \\ & + \frac{c^2\hbar^2}{(2\mu c^2 - B_\Lambda - D_-)^2} \Bigg[-\frac{n^2R^3}{2} j_{l-2}(nR) j_l(nR) \\ & + \frac{n^2R^3}{2} j_{l-1}^2(nR) - \frac{(k-l)^2}{2l+1} R j_l^2(nR) \Bigg] \\ & + \frac{c^2\hbar^2}{(2\mu c^2 - B_\Lambda)^2} \frac{j_l^2(nR)}{k_l^2(n_0R)} \Bigg[\frac{n_0^2R^3}{2} k_{l-2}(n_0R) k_l(n_0R) \\ & - \frac{n_0^2R^3}{2} k_{l-1}^2(n_0R) + \frac{(k-l)^2}{2l+1} R k_l^2(n_0R) \Bigg] \Bigg\}^{-1/2} \end{aligned} \tag{10}$$

It is interesting to note that the normalization constant in the nonrelativistic limit, that is, omitting terms of the order $(\mu c^2)^{-1}$ and higher, is reduced to the expression

$$\tilde{N}_{nr} = \frac{1}{n} \frac{2^{1/2}}{R^{3/2}} \left[\frac{j_l^2(nR)}{k_l^2(n_0R)} k_{l-1}(n_0R) k_{l+1}(n_0R) - j_{l-1}(nR) j_{l+1}(nR) \right]^{-1/2} \tag{11}$$

given by Sitenko and Tartakovskii [14] for the nonrelativistic square-well case (where the n and n_0 in this formula are considered in the nonrelativistic limit).

The root-mean-square radii of the Λ-particle orbits in hypernuclei for every bound state are obtained by means of the expression

$$\langle r^2\rangle^{1/2} = \left\{\int_0^\infty r^2[G^2(r) + F^2(r)]\, dr\right\}^{1/2} \tag{12}$$

since G and F are normalized by means of condition (9). Calculating the above integrals and using expression (10) for the normalization constant, we derive the following lengthy formula for $\langle r^2\rangle^{1/2}$:

$$\begin{aligned}
\langle r^2\rangle^{1/2} = \frac{R}{3^{1/2}}\Bigg[&\frac{(2l+3)(2l-1)}{2n^2R^2}\left[-j_{l-1}(nR)j_{l+1}(nR)+\frac{1}{2}j_l^2(nR)\right]\\
&+\left\{\frac{1}{2}[j_{l-1}(nR)-j_{l+1}(nR)]-\frac{j_l(nR)}{nR}\right\}^2\\
&+\frac{D_+ - B_\Lambda}{2\mu c^2[1-(B_\Lambda/2\mu c^2)-(D_-/2\mu c^2)]}\\
&\times\Bigg(\frac{(2l+1)(2l-3)}{2n^2R^2}\left[-j_{l-2}(nR)j_l(nR)+\frac{1}{2}j_{l-1}^2(nR)\right]\\
&+\left\{\frac{1}{2}[j_{l-2}(nR)-j_l(nR)]-\frac{j_{l-1}(nR)}{nR}\right\}^2+j_{l-1}^2(nR)\\
&+\frac{3(k-l)^2}{n^2R^2}[-j_{l-1}(nR)j_{l+1}(nR)+j_l^2(nR)]\\
&+\frac{3(k-l)}{4n^2R^2}\Bigg\{(2l+1)[j_{l-2}(nR)-j_l(nR)]^2\\
&+\left(4-\frac{(2l-1)(2l+1)}{n^2R^2}\right)(2l-1)j_{l-1}^2(nR)\Bigg\}\Bigg)\\
&+\frac{j_l^2(nR)}{k_l^2(n_0R)}\Bigg(\frac{(2l+3)(2l-1)}{2n_0^2R^2}\left[-k_{l-1}(n_0R)\,k_{l+1}(n_0R)+\frac{1}{2}k_l^2(n_0R)\right]\\
&+\left\{\frac{1}{2}[k_{l-1}(n_0R)+k_{l+1}(n_0R)]+\frac{k_l(n_0R)}{n_0R}\right\}^2\Bigg)\\
&+\frac{B_\Lambda}{2\mu c^2[1-(B_\Lambda/2\mu c^2)]}\frac{j_l^2(nR)}{k_l^2(n_0R)}\Bigg(\frac{(2l+1)(2l-3)}{2n_0^2R^2}\Bigg[-k_{l-2}(n_0R)k_l(n_0R)\\
&+\frac{1}{2}k_{l-1}^2(n_0R)\Bigg]+\left\{\frac{1}{2}[k_{l-2}(n_0R)+k_l(n_0R]+\frac{k_{l-1}(n_0R)}{n_0R}\right\}^2
\end{aligned}$$

$$
\begin{aligned}
&- k_{l-1}^2(n_0R) + \frac{3(k-l)^2}{n_0^2R^2}[k_{l-1}(n_0R)\,k_{l+1}(n_0R) - k_l^2(n_0R)] \\
&- \frac{3(k-l)}{4n_0^2R^2}\Bigg\{(2l+1)[k_{l-2}(n_0R) + k_l(n_0R)]^2 \\
&- \left(4 + \frac{(2l-1)(2l+1)}{n_0^2R^2}\right)(2l-1)k_{l-1}^2(n_0R)\Bigg\}\Bigg)\Bigg]^{1/2} \\
&\times \Bigg\{\frac{j_l^2(nR)}{k_l^2(n_0R)}k_{l-1}(n_0R)k_{l+1}(n_0R) - j_{l-1}(nR)j_{l+1}(nR) \\
&+ \frac{D_+ - B_\Lambda}{2\mu c^2[1-(B_\Lambda/2\mu c^2)-(D_-/2\mu c^2)]}\Bigg[-j_{l-2}(nR)j_l(nR) \\
&+ j_{l-1}^2(nR) - \frac{(k-l)^2}{2l+1}\frac{2}{n^2R^2}j_l^2(nR)\Bigg] \\
&+ \frac{B_\Lambda}{2\mu c^2[1-(B_\Lambda/2\mu c^2)]}\frac{j_l^2(nR)}{k_l^2(n_0R)}\Bigg[k_{l-2}(n_0R)k_l(n_0R) - k_{l-1}^2(n_0R) \\
&+ \frac{(k-l)^2}{2l+1}\frac{2k_l^2(n_0R)}{n_0^2R^2}\Bigg]\Bigg\}^{-1/2} \qquad (13)
\end{aligned}
$$

In view of the complexity of the above expression, approximate formulas for the root-mean-square radii of the Λ-particle orbits in hypernuclei for every bound state were obtained by using the following asymptotic forms for the spherical Bessel and MacDonald functions:

$$j_l(x) \simeq \frac{1}{x}\cos\left[x - \frac{1}{2}(l+1)\pi\right] \qquad (14a)$$

$$k_l(x) \simeq \frac{1}{x}\frac{\pi}{2}e^{-x} \qquad (14b)$$

and ignoring terms which are expected not to be significant, such as terms involving products of the small quantities

$$\frac{B_\Lambda}{2\mu c^2[1-(B_\Lambda/2\mu c^2)]}, \qquad \frac{D_+ - B_\Lambda}{2\mu c^2[1-(B_\Lambda/2\mu c^2) - D_-/2\mu c^2)]}$$

etc.

Thus, from (13) one obtains the simpler expression

$$\langle r^2\rangle^{1/2} \simeq \frac{R}{3^{1/2}} + \frac{1}{2n_0\,3^{1/2}} - \frac{1}{2n_0\,3^{1/2}}\left[\cos(2nR - l\pi) + \sin(2nR - l\pi)\frac{n_0}{n}\right] \quad (15)$$

If we set in this expression

$$\varphi = \operatorname{arccot}\frac{n_0}{n} \quad (16)$$

and express R as a function of A_c ($R = r_0 A_c^{1/3}$) in the first term, we have

$$\langle r^2\rangle^{1/2} \simeq \frac{r_0 A_c^{1/3}}{3^{1/2}} + \frac{1}{2n_0 3^{1/2}} - \frac{1}{2n_0 3^{1/2}}\frac{\sin(2nR - l\pi + \varphi)}{\sin\varphi} \quad (17)$$

Observing that the ratio

$$\frac{\sin(2nR - l\pi + \varphi)}{\sin\varphi} \quad (18)$$

can be approximated by a constant $C_{N,l}$ (where N is the principal quantum number $N = 1, 2, \ldots$), one obtains the simple approximate formula

$$\langle r^2\rangle^{1/2} \simeq \frac{r_0}{3^{1/2}} A_c^{1/3} + \frac{(1 - C_{N,l})}{2n_0 3^{1/2}} \quad (19)$$

for the lower bound states. From this formula one can deduce immediately the almost linear behavior of the curves $\langle r^2\rangle^{1/2}$ versus $A_c^{1/3}$ for the heavier hypernuclei. (In formula (19), if $l = 0$, $C_{1,0} = -1.2$; if $l = 1$, $C_{1,1} = -1.4$, etc.)

Other approximate expressions can also be derived from (15).

In the ground state ($l = 0$) all the expressions for the root-mean-square radii go over to those given in ref. 13.

We have also calculated the potential energy of the Λ particle in every bound state. The potential energy operator is of the form

$$V(r) = \beta U_S(r) + U_V(r) \quad (20)$$

The expectation value of $V(r)$ for the rectangular potentials considered here is

$$\langle V(r)\rangle = -D_+\int_0^R G^2(r)\,dr + D_-\int_0^R F^2(r)\,dr \quad (21)$$

Using expressions (4), (5), and (10), we find for the potential energy the expression

$$\langle V(r)\rangle = \Bigg\{ -D_+[-j_{l-1}(nR)j_{l+1}(nR) + j_l^2(nR)]$$

$$+ D_-\frac{D_+ - B_\Lambda}{2\mu c^2[1 - (B_\Lambda/2\mu c^2) - (D_-/2\mu c^2)]}$$

$$\times \left[-j_{l-2}(nR)j_l(nR) + j_{l-1}^2(nR) - \frac{(k-l)^2}{2l+1}\frac{2}{n^2R^2} j_l^2(nR) \right] \Bigg\}$$

$$\times \left\{ \frac{j_l^2(nR)}{k_l^2(n_0R)} k_{l-1}(n_0R)k_{l+1}(n_0R) - j_{l-1}(nR)j_{l+1}(nR) \right.$$

$$+ \frac{D_+ - B_\Lambda}{2\mu c^2[1 - (B_\Lambda/2\mu c^2) - (D_-/2\mu c^2)]}$$

$$\times \left[-j_{l-2}(nR)j_l(nR) + j_{l-1}^2(nR) - \frac{(k-l)^2}{2l+1}\frac{2}{n^2R^2} j_l^2(nR) \right]$$

$$+ \frac{B_\Lambda}{2\mu c^2[1 - (B_\Lambda/2\mu c^2)]} \frac{j_l^2(nR)}{k_l^2(n_0R)} \left[k_{l-2}(n_0R)k_l(n_0R) - k_{l-1}^2(n_0R) \right.$$

$$\left.\left. + \frac{(k-l)^2}{2l+1}\frac{2}{n_0^2R^2} k_l^2(n_0R) \right] \right\}^{-1} \tag{22}$$

In the nonrelativistic limit the above expression goes over to the expression

$$\langle V(r) \rangle = -D_+ + \left\{ D_+ j_l^2(nR) \left[\frac{k_{l-1}(n_0R)k_{l+1}(n_0R)}{k_l^2(n_0R)} - 1 \right] \right\}$$

$$\times \left\{ -j_{l-1}(nR)j_{l+1}(nR) + j_l^2(nR) \frac{k_{l-1}(n_0R)k_{l+1}(n_0R)}{k_l^2(n_0R)} \right\}^{-1} \tag{23}$$

From expression (22) one can obtain also the following simple approximate expression for the potential energy:

$$\langle V(r) \rangle \simeq -D_+ + (D_+ + D_-)(D_+ - B_\Lambda)(2\mu c^2 - 2B_\Lambda - D_- + D_+)^{-1} \tag{24}$$

This is expected to hold, however, for very large values of A_c (and the lowest state).

3. NUMERICAL RESULTS AND COMMENTS

In this section we present our results concerning the root-mean-square radii of the Λ-particle orbits in hypernuclei and also its potential energies. These results were obtained using expressions (13), (15), (22), and the following potential parameters:

$$D_- = 300 \text{ MeV (fixed)}, \qquad D_+ = 25.74 \text{ MeV}, \qquad r_0 = 1.22 \text{ fm}$$

derived from an "overall fit," that is, a least squares fit to the experimental binding energies B_Λ of all states for a number of hypernuclei. In the above fit the potential parameter D_- was kept fixed to the value $D_- = 300$ MeV [11]. Also, for μ the Λ-core reduced mass was used.

The results for the root-mean-square radii of the Λ-particle orbits in its ground and excited states are given in Tables I and II. In Table I the results derived with the exact expression (13) are given for a number of states, while in Table II those derived with the approximate expression (15) are tabulated for the lower states.

The results for the potential energy of the Λ-particle in the ground and excited states, calculated using the exact expression (22), are given in Table III. It is seen that the variation of $|\langle V(r)\rangle|$ with A_c is increasing, as expected.

In Fig. 1 the results given in Table I (i.e., the $\langle r_\Lambda^2\rangle^{1/2}$) are plotted versus $A_c^{1/3}$. It is interesting to note that the behavior is rather similar to that reported in ref. 15. The values obtained with the present approach and the above values of the parameters are larger than the corresponding ones of that reference. Note also that the almost linear behavior of the curves $\langle r_\Lambda^2\rangle^{1/2}$ versus $A_c^{1/3}$ for the heavier hypernuclei is fairly well understood on the basis of expression (19), which predicts such a behavior for large A_c.

Some additional remarks are in order.

First, it would be of interest to compare the theoretical results with corresponding experimental values. This can be done only for Λ-binding energies in single-particle states for certain hypernuclei for which experimental B_Λ values $B^{\exp}$ are available. Even in these cases the experimental errors

Table I. Root-Mean-Square Radii $\langle r_\Lambda^2\rangle^{1/2}$ of the Λ-Particle Orbits in the Ground and Excited States for Various Hypernuclei Obtained Semianalytically Using Expression (13)[a]

A_c	$s_{1/2}$ (fm)	$p_{3/2}$ (fm)	$p_{1/2}$ (fm)	$d_{5/2}$ (fm)	$d_{3/2}$ (fm)	$f_{7/2}$ (fm)	$f_{5/2}$ (fm)
8	2.27						
10	2.27						
11	2.28						
12	2.30						
15	2.35	3.44	3.96				
27	2.61	3.20	3.17				
31	2.68	3.26	3.22				
39	2.83	3.38	3.33	4.06	4.26		
50	3.00	3.54	3.50	4.05	4.02		
88	3.46	4.04	4.00	4.47	4.40	4.9	4.86
137	3.90	4.54	4.49	4.97	4.90	5.32	5.24
207	4.39	5.08	5.04	5.54	5.48	5.89	5.81

[a] The values of the potential parameters are $r_0 = 1.22$ fm, $D_+ = 25.74$ MeV, $D_- = 300$ MeV.

Table II. Root-Mean-Square Radii $\langle r_\Lambda^2 \rangle^{1/2}$ of the Λ-Particle Orbits in the Ground and First Excited States for Various Hypernuclei Obtained Using the Approximate Expression (15)[a]

A_c	$s_{1/2}$ (fm)	$p_{3/2}$ (fm)	$p_{1/2}$ (fm)
8	2.38		
10	2.40		
11	2.42		
12	2.44		
15	2.51	3.56	4.61
27	2.80	3.12	3.16
31	2.89	3.16	3.18
39	3.05	3.28	3.28
50	3.24	3.44	3.43
88	3.75	3.93	3.92
137	4.23	4.42	4.41
207	4.75	4.96	4.95

[a] The values of the potential parameters used are the same as in Table I.

are not always very small, because of resolution problems. Unfortunately, there are no experimental data for the root-mean-square radii for the Λ-particle orbits. For that reason theoretical estimates for these quantities should be considered rather valuable since there is no other source of information for them. It should be noted, however, that a comparison can be made between the calculated values of $\langle r^2 \rangle^{1/2}$, $\langle r^2 \rangle^{1/2}(B^{\mathrm{theor}})$, and the $\langle r^2 \rangle^{1/2}(B^{\mathrm{exp}})$, that is,

Table III. Potential Energies $\langle V \rangle$ of the Λ Particle in the Ground and Excited States for Various Hypernuclei Obtained Using Expression (22).

A_c	$s_{1/2}$ (MeV)	$p_{3/2}$ (MeV)	$p_{1/2}$ (MeV)	$d_{5/2}$ (MeV)	$d_{3/2}$ (MeV)	$f_{7/2}$ (MeV)	$f_{5/2}$ (MeV)
8	−17.44						
10	−18.82						
11	−19.32						
12	−19.75						
15	−20.70	−13.46	−11.03				
27	−22.64	−18.66	−17.91				
31	−22.81	−19.42	−18.81				
39	−23.26	−20.47	−20.03	−15.89	−14.14		
50	−23.67	−21.38	−21.07	−18.02	−17.04		
88	−24.35	−22.87	−22.72	−20.88	−20.47	−18.11	−17.12
137	−24.72	−23.65	−23.57	−22.26	−22.03	−20.48	−19.99
207	−24.98	−24.18	−24.13	−23.16	−23.02	−21.9	−21.62

[a] The values of the parameters are the same as in Table I.

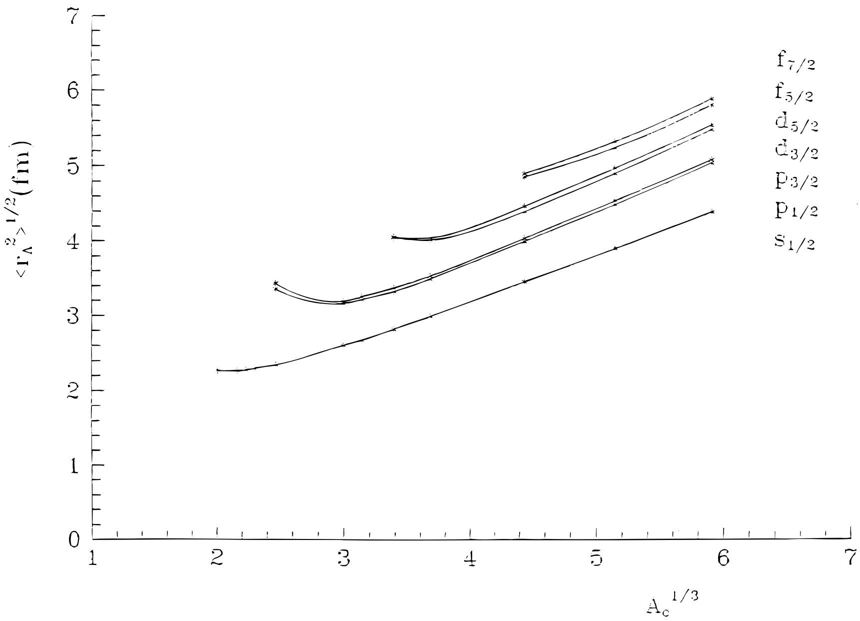

Fig. 1. Variation of the $\langle r_\Lambda^2\rangle^{1/2}$ with $A_c^{1/3}$ for the ground and first excited states of Λ hypernuclei.

between the root-mean-square radii obtained by using for B_Λ[which enters in the analytic expression (13) either explicitly or through n and n_0—see expressions (7) and (8)] the theoretical values B^{theor} or the experimental binding energy values B^{exp}. We concentrate here on Λ in its ground state; the relevant results are displayed in Table IV. It is seen that the agreement between B^{theor} and B^{exp} and between $\langle r^2\rangle^{1/2}(B^{\text{theor}})$ and $\langle r^2\rangle^{1/2}$ (B^{exp}) is usually

Table IV. Comparison of 1*s* Λ-Particle Binding Energies and Root-Mean-Square Radii of Its Orbit for Various Hypernuclei[a]

A_c	Hypernucleus	B_{1s}^{theor}	B_{1s}^{exp}	$\langle r^2\rangle_{1s}^{1/2}(B^{\text{theor}})$	$\langle r^2\rangle_{1s}^{1/2}(B^{\text{exp}})$
8	${}^{9}_{\Lambda}$Be	8.36	6.49 ± 0.68	2.27	2.29 ± 0.02
10	${}^{11}_{\Lambda}$B	10.10	10.1 ± 0.1	2.27	2.27 ± 0.00
11	${}^{12}_{\Lambda}$C	10.81	10.75 ± 0.1	2.28	2.28 ± 0.00
12	${}^{13}_{\Lambda}$C	11.43	11.69 ± 0.1	2.30	2.30 ± 0.00
15	${}^{16}_{\Lambda}$O	12.94	12.5 ± 0.35	2.35	2.33 ± 0.02
27	${}^{28}_{\Lambda}$Si	16.31	16.0 ± 0.29	2.61	2.57 ± 0.03
31	${}^{32}_{\Lambda}$S	16.98	17.5 ± 0.5	2.68	2.74 ± 0.05
39	${}^{40}_{\Lambda}$Ca	18.00	18.7 ± 1.1	2.83	2.92 ± 0.15
50	${}^{51}_{\Lambda}$V	18.99	19.9 ± 1.0	3.00	3.15 ± 0.17
88	${}^{89}_{\Lambda}$Y	20.82	22.1 ± 1.6	3.46	3.83 ± 0.43

[a] See text for notation.

Table V. Exact and Approximate Values of $\langle r^2\rangle^{1/2}/R|_{s_{1/2}}$ and Their Differences for Various Values of s^{-1} (See Text)

s^{-1}	$\langle r^2\rangle^{1/2}/R|_{s_{1/2}}$ Exact	Approximate	Difference	Percentage difference
2.38	1.05	1.07	0.02	1.9
2.50	0.98	1.01	0.03	3.1
2.63	0.93	0.97	0.04	4.3
2.77	0.88	0.93	0.05	5.7
3.02	0.82	0.87	0.05	6.1
3.13	0.80	0.85	0.05	6.3
3.24	0.79	0.84	0.05	6.4
3.52	0.75	0.80	0.05	6.7
4.35	0.69	0.74	0.05	7.2
4.57	0.68	0.73	0.05	7.4
4.95	0.66	0.71	0.05	7.6
5.40	0.65	0.70	0.05	7.7
6.55	0.62	0.67	0.05	8.1
7.61	0.61	0.66	0.05	8.2
8.74	0.60	0.65	0.05	8.3
10.0	0.59	0.64	0.05	8.5
12.50	0.58	0.63	0.05	8.6
16.67	0.56	0.61	0.05	8.9
25.00	0.55	0.60	0.05	9.0

fairly satisfactory. Similar conclusions may be drawn for Λ-excited states such as the *p*, *d*, *f* ones.

Finally, it would be of interest to discuss also the validity of the approximations in connection with the range of parameters. Due, however, to the existence of four parameters in the model, namely the mass μ, the two potential depths D_+, D_-, and the common potential radius R, such a discussion would be rather complicated. It seems advisable, therefore, to consider for this discussion the nonrelativistic limit, which leads to a useful simplification. This should be adequate as long as the relativistic effects are sufficiently small, which is often the case. In that limit, one can see from expressions (6), (13), and (15) that the corresponding dimensionless quantities B_Λ/D_+ and $\langle r^2\rangle^{1/2}/R$ depend only on a single (dimensionless) parameter $s^{-1} = (2\mu D_+ R^2/\hbar^2)^{1/2}$. We can therefore estimate the errors made in calculating $\langle r^2\rangle^{1/2}/R$ through the approximate expression (15) instead of the corresponding exact one, by calculating the percentage differences. The results of Table V show the corresponding values of $\langle r^2\rangle^{1/2}/R$ (using for B_Λ in (15) expression (25) of ref. 12) and the percentage differences for various values of s^{-1} considering the 1*s* state. It is seen that these differences are of the order of 2–9% for a

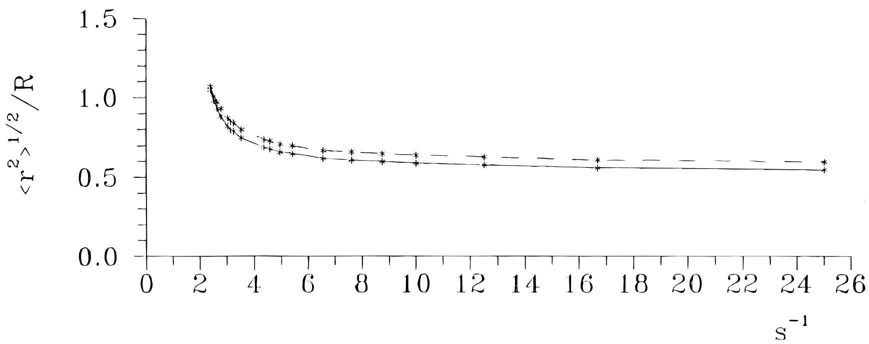

Fig. 2. Variation of $\langle r^2\rangle^{1/2}/R$ for the 1*s* state with s^{-1} (see text). The exact results are given by the solid line and the approximate ones by the dashed line.

rather wide range of s^{-1}, namely, $2.38 \leq s^{-1} \leq 25.0$. These results are also depicted in Fig. 2.

ACKNOWLEDGMENT

Partial financial support from the Greek General Secretariat of Research and Technology is gratefully acknowledged.

REFERENCES

1. A. Das and A. C. Melissinos, *Quantum Mechanics: A Modern Introduction*, Gordon and Breach, New York (1988).
2. L. Ballentine, *Quantum Mechanics*, Prentice-Hall, Englewood Cliffs, New Jersey (1990).
3. D. Park, *Introduction to Quantum Theory*, 3rd ed., McGraw-Hill, New York (1992).
4. S. Gasiorowicz, *Quantum Physics*, Wiley, New York (1974).
5. C. B. Daskaloyannis, M. E. Grypeos, C. G. Koutroulos, S. E. Massen, and D. S. Saloupis, *Phys. Lett.* **134B** (1984), 147.
6. H. D. Duerr, *Phys. Rev.* **103** (1956) 469; *Phys. Rev.* **109** (1958) 1347.
7. J. D. Walecka, *Ann. Phys.* **83** (1974) 491.
8. L. N. Savushkin and V. N. Fomenko, *Sov. J. Part. Nucl.* **8** (1977) 371; L. N. Savushkin, *Sov. J. Nucl. Phys.* **30** (1979) 340.
9. A. Bouyssy, S. Marcos, and J. F. Mathiot, *Nucl. Phys. A* **415** (1984) 497.
10. C. G. Koutroulos and M. E. Grypeos, *Phys. Rev. C* **40** (1989), 275.
11. G. J. Papadopoulos, C. G. Koutroulos, and M. E. Grypeos, *Int. J. Mod. Phys. E* **3** (1994) 939.
12. M. E. Grypeos, C. G. Koutroulos, and G. J. Papadopoulos, *Phys. Rev. A* **50** (1994) 29
13. C. G. Koutroulos and G. J. Papadopoulos, *Prog. Theor. Phys.* **90** (1993) 1039
14. A. G. Sitenko and V. K. Tartakovskii, *Lectures on the Theory of the Nucleus*, Pergamon Press, Oxford (1975).
15. M. Rayet, *Ann. Phys.* **102** (1976) 226.